

On The Scattering Process in Quantum Optics

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(Dated: September 12, 2014)

The derivation of a quantum Markovian model for an opto-mechanical system consisting of a quantum mechanical mirror interacting with quantum optical input fields via radiation pressure is difficult problem which ultimately involves the scattering process of quantum stochastic calculus. We show that while the scattering process may be approximated in a singular limit by regular processes using different schemes, however the limit model is highly sensitive to how the approximation scheme is interpreted mathematically. We find two main types of stochastic limits of regular models, and illustrate the origin of this difference at the level of one particle scattering. As an alternative modelling scheme, we consider models of mirrors as non-trivial dielectric medium with a boundary that is itself quantized. Rather than treating the plane waves for the electromagnetic field, we take the actual physical modes and quantize these. The input-output formalism is then obtained in the far zone where the plane wave approximation is valid. Several examples are considered, and the quantum stochastic model is derived. We also consider the quantum trajectories problem for continual measurement of the reflect output fields, and derive the stochastic master equations for homodyning and photon counting detection to estimate the mirror observables.

PACS numbers: 03.65.Nk, 42.50.Dv, 03.65.Db, 02.30.Mv

I. INTRODUCTION

The theory of quantum stochastic calculus of Hudson and Parthasarathy [1] has a long history of applications to quantum open systems. In addition to introducing quantum stochastic integrals with respect to creation and annihilation processes, they included integration with respect to scattering processes. While not originally motivated by quantum input-output models, this additional feature allowed unitary rotations of input fields to be considered in conjunction to displacements. This has lead to a unified treatment of state-based input-output models that has been exploited in model interconnections of such models into quantum feedback networks, and in particular to rigorously model simplification and reduction procedures such as adiabatic elimination of open systems [2]-[6]; other areas of application include the qubit limit of cQED [7].

The scattering operator S appearing in the unitary quantum stochastic differential equations (QSDE) however has attracted some comparison to S -matrix of usual scattering theory, however, its origin and role is rather different. The scattering processes are also less familiar to the Physics community than the usual creation annihilation processes, as they were not part of the original input/output formalism of quantum optics [8],[9].

Typically quantum stochastic processes obeying a non-trivial Itô table arise as singular limits of approximating regular processes. This is a delicate problem for classical stochastic processes, however there are additional issues related to approximating the scattering processes as different schemes for otherwise similar dynamics lead to different limit evolutions.

For an m input model we have quantum white noise input process $b_j(t)$ for $j = 1, \dots, m$ satisfying singular commutation relations $[b_j(t), b_k^\dagger(s)] = \delta_{jk}\delta(t-s)$ and we define the $(m+1)^2$ fundamental processes [1]

$$\Lambda_{\alpha\beta}(t) = \int_0^t b_\alpha^\dagger(s)b_\beta(s)ds \quad (1)$$

where we also include the index 0 by setting $b_0(t) \equiv 1$. In this way $\Lambda_{00}(t) = t$, while $B_j(t) := \Lambda_{0k}(t) = \int_0^t b_k(s)ds$ and $B_j^\dagger(t) := \Lambda_{j0}(t) = \int_0^t b_j^\dagger(s)ds$ are the processes of annihilation and creation. $\Lambda_{jk}(t)$ describes the process where a quanta in channel k is annihilated and another immediately created in channel j at some time in the interval $[0, t]$. The $\Lambda_{\alpha\beta}(t)$ are well-defined operators acting on the Fock space \mathfrak{F} over \mathbb{C}^m -valued square-integrable functions of positive time $t \geq 0$. We note the quantum Itô table [1]

$$d\Lambda_{\alpha\beta}(t)d\Lambda_{\mu\nu}(t) \equiv \hat{\delta}_{\beta\mu}d\Lambda_{\alpha\nu}(t) \quad (2)$$

where $\hat{\delta}_{\beta\nu} = 1$ if $\beta = \nu \neq 0$, and vanishes otherwise.

Let us fix a system with Hilbert space \mathfrak{h} . The quantum stochastic differential equation (QSDE) on $\mathfrak{h} \otimes \mathfrak{F}$ (implied sum over repeated Greek indices from 0 to m)

$$dU(t) = G_{\alpha\beta} \otimes d\Lambda_{\alpha\beta}(t) U(t), \quad (3)$$

with initial condition $U(0) = I$, possesses a unique solution for bounded operators $G_{\alpha\beta}$ on \mathfrak{h} . The necessary and sufficient conditions for the process $U(t)$ to be unitary are that (implied sum over repeated Latin indices from 1 to m) [1]

$$\begin{aligned} G_{jk} &= S_{jk} - \delta_{jk}, G_{j0} = L_j, \\ G_{0k} &= -L_k^\dagger S_{lk}, G_{00} = -\frac{1}{2}L_l^\dagger L_l - iH \end{aligned} \quad (4)$$

with $S = [S_{jk}]$ unitary, $L = [L_j]$ arbitrary, and H self-adjoint. The triple (S, L, H) then determines the model.

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A. Approximations by Regularized Hamiltonians

1. Scheme #1

In practice, the singular processes are idealisations. However, working in the same Fock space, it is possible to approximate the fundamental processes by regular ones obtained by smearing with some mollifying function. In the following, we will denote by δ_n a regular function with compact support parametrized by $n > 0$ and converging to a delta function as $n \rightarrow \infty$. For definiteness we may fix an integrable function g with support $[-c, c]$ (i.e., $g(x) = 0$ for $|x| > c$) for some finite range $c > 0$, such that $g(-x) = g(x)$ and $\int_{-c}^c g(x) dx = 1$. We may then take $\delta_n(x) = n g(nx)$, and these vanish outside the interval $[-c/n, c/n]$.

We shall now describe two possible schemes to formally approximate $b_\alpha^\dagger(t)b_\beta(t)$.

Let $E_{\alpha\beta}$ be a collection of operators on the Hilbert space \mathfrak{h} of a fixed system such that $E_{\alpha\beta}^\dagger = E_{\beta\alpha}$. For convenience we assume that they are bounded. We write $E_{\ell\ell}$ for the matrix $[E_{jk}]$, $E_{\ell 0}$ for the column vector $[E_{j0}]$, and $E_{0\ell}$ for the row vector $[E_{0k}]$.

The matrix $E_{\ell\ell}$ will be called the *exchange matrix*. In principle, E_{jk} gives the strength of the interaction causing an input quantum of type k to be annihilated and replace with a quantum of type k .

We set $\tilde{\lambda}_{\alpha\beta}^{(n)}(t) = \int \delta_n(t-s) b_\alpha^\dagger(s) b_\beta(s) ds$, or more exactly

$$\tilde{\lambda}_{\alpha\beta}^{(n)}(t) = \int \delta_n(t-s) d\Lambda_{\alpha\beta}(s), \quad (5)$$

$$\tilde{b}_k^{(n)}(t) = \int \delta_n(t-s) dB_k(s). \quad (6)$$

A unitary $\tilde{U}^{(n)}(t)$ is defined as the solution to the Schrödinger equation with time-dependent Hamiltonian

$$\begin{aligned} \tilde{H}^{(n)}(t) &= \int \delta_n(t-s) E_{\alpha\beta} \otimes d\Lambda_{\alpha\beta}(s), \\ &= E_{\alpha\beta} \otimes \tilde{\lambda}_{\alpha\beta}^{(n)}(t) \\ &= E_{jk} \otimes \tilde{\lambda}_{jk}^{(n)}(t) \\ &\quad + E_{j0} \otimes \tilde{b}_j^{(n)\dagger}(t) + E_{0k} \otimes \tilde{b}_k^{(n)}(t) + E_{00}. \end{aligned} \quad (7)$$

The limit process $\tilde{U}(t)$ then exists, is unitary and described by the triple

$$\tilde{S} = e^{-iE_{\ell\ell}}, \quad \tilde{L} = \frac{e^{-iE_{\ell\ell}} - 1}{E_{\ell\ell}} E_{\ell 0}, \quad \tilde{H} = E_{00} - E_{01} \frac{E_{\ell\ell} - \sin(E_{\ell\ell})}{(E_{\ell\ell})^2} E_{\ell 0}. \quad (8)$$

The limit is best understood as a trotterized time-ordered exponential introduced by Hølevø [10]

$$\tilde{U}(t) = \vec{T}_H e^{-i \int_0^t dE} := \lim_{\max |t_{k+1} - t_k| \rightarrow 0} e^{-iE(t_N, t_{N-1})} \dots e^{-iE(t_2, t_1)} e^{-iE(t_1, t_0)} \quad (9)$$

where $E(t_2, t_1) = \int_{t_1}^{t_2} E_{\alpha\beta} \otimes d\Lambda_{\alpha\beta}(s) \equiv E_{\alpha\beta} \otimes \{\Lambda_{\alpha\beta}(t_2) - \Lambda_{\alpha\beta}(t_1)\}$ and $t = t_N > \dots > t_1 > t_0 = 0$.

The relationship between the coefficients is $\tilde{G}_{\alpha\beta} \otimes d\Lambda_{\alpha\beta} \equiv e^{-iE_{\alpha\beta} \otimes d\Lambda_{\alpha\beta}} - 1$.

A unitary $U^{(n)}(t)$ is defined as the solution to the Schrödinger equation with time-dependent Hamiltonian

2. Scheme #2

We alternatively set

$$\lambda_{jk}^{(n)}(t) = \tilde{b}_j^{(n)\dagger}(t) \tilde{b}_k^{(n)}(t) \quad (10)$$

and $\lambda_{j0}^{(n)}(t) = \tilde{b}_j^{(n)\dagger}(t)$, $\lambda_{0k}^{(n)}(t) = \tilde{b}_k^{(n)}(t)$, $\lambda_{00}^{(n)}(t) = 1$.

$$\begin{aligned} H^{(n)}(t) &= E_{\alpha\beta} \otimes \lambda_{\alpha\beta}^{(n)}(t) \\ &= E_{jk} \otimes \tilde{b}_j^{(n)\dagger}(t) \tilde{b}_k^{(n)}(t) \\ &\quad + E_{j0} \otimes \tilde{b}_j^{(n)\dagger}(t) + E_{0k} \otimes \tilde{b}_k^{(n)}(t) + E_{00}. \end{aligned} \quad (11)$$

The limit process $U(t)$ exists and is then described by the triple [12]

$$S = \frac{I - \frac{i}{2} E_{\ell\ell}}{I + \frac{i}{2} E_{\ell\ell}}, \quad L = \frac{-i}{I + \frac{i}{2} E_{\ell\ell}} E_{\ell 0}, \quad H = E_{00} + \frac{1}{2} E_{0\ell} \text{Im} \left\{ \frac{I}{I + \frac{i}{2} E_{\ell\ell}} \right\} E_{\ell 0}. \quad (12)$$

The limit process is best understood as the Stratonovich (or symmetric) integral [11]

$$U(t) = 1 - i \int_0^t dE(s) \circ U(s) \equiv 1 - i \lim_{\max |t_{k+1} - t_k| \rightarrow 0} \sum_k E(t_{k+1}, t_k) U\left(\frac{t_{k+1} + t_k}{2}\right) \quad (13)$$

using a midpoint rule for sample the integrand.

The Stratonovich unitary may be denoted as

$$U(t) = \vec{T}_D e^{-i \int_0^t dE} \quad (14)$$

and formally the ordering \vec{T}_D is the Dyson chronological ordering of the white noise operators $b_j(t)$ and $b_j^\dagger(t)$, [12]

II. SINGLE PARTICLE SCATTERING MODELS

The limit procedures above are technically involved, however, we can obtain some insight into what is going on at a simpler level. We consider the situation of a quantum particle moving along the x -axis with Hamiltonian

$$H = -p + V \quad (15)$$

The free part $-p = i\partial$ described propagation at unit speed down the axis, while the potential V is localized in some region about the origin, see Fig. 1.

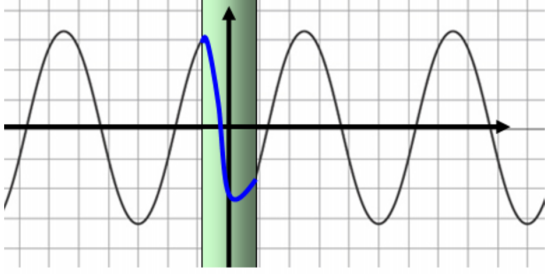


FIG. 1. (Color online) A solution to (15) with a region about the origin where $V \neq 0$, and plane wave behaviour outside.

If the potential is to be modeled as exactly localized at the origin - say a delta potential $V = \varepsilon\delta(x)$. then we are forced to consider the operator $i\partial$ for $x \neq 0$ but this is not a self-adjoint operator. In this case the corresponding H must be a self-adjoint extension of $i\partial$ on the punctured line and it is well known that its domain is the set of all functions ψ with derivative $\psi'(x)$ well-defined for $x \neq 0$ and $(\int_{-\infty}^0 + \int_0^{\infty}) (|\psi(x)|^2 + |\psi'(x)|^2) dx < \infty$ satisfying a boundary condition (see section X.1 of Reed and Simon [13], volume 2, especially example 1)

$$\psi(0^-) = s\psi(0^+) \quad (16)$$

where $s = e^{i\theta}$ is a unimodular complex number. See Fig. 2.

Specifically, we have the integration by parts formula

$$\begin{aligned} & \left(\int_{-\infty}^{0^-} + \int_{0^+}^{\infty} \right) \phi^*(x) \psi'(x) dx = \\ & \phi^*(0^-) \psi(0^-) - \phi^*(0^+) \psi(0^+) \\ & - \left(\int_{-\infty}^{0^-} + \int_{0^+}^{\infty} \right) \phi^{*'}(x) \psi(x) dx, \end{aligned}$$

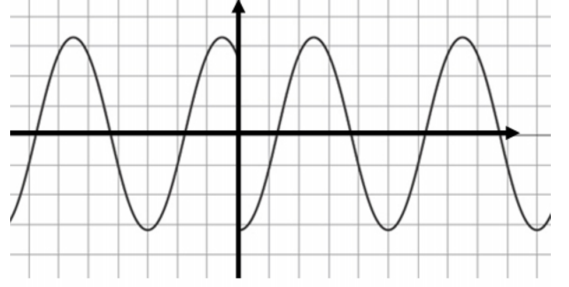


FIG. 2. (Color online) The limit of a singular potential. A plane wave solution with a phase jump s at the boundary.

and the boundary term vanishes exactly if both ϕ and ψ satisfy the (same!) boundary condition (16) with s unitary.

As s is arbitrary, we have an infinity of possible self-adjoint extensions H_s . Ultimately the choice of s comes down to physical modeling.

We shall look at two different approximation schemes now.

A. When the phase jump will be an exponential of the coupling parameter.

We take the potential to have the form

$$V = \delta_n(x) \quad (17)$$

We consider a stationary state with far field behaviour $\psi(x) = A_{\pm} e^{-ikx}$ for $x \rightarrow \pm\infty$. We have $i\psi'(x) + V(x)\psi(x) = E\psi(x)$. Away from the origin, where the potential is zero, we see that $E \equiv k$. We can integrate to get

$$\psi(b) e^{ikb} e^{-i \int_a^b V(x) dx} = \psi(a) e^{ika}.$$

We have $V(x) = \varepsilon\delta_n(x)$, and so taking $n \rightarrow \infty$ $\psi(b) e^{ikb} e^{i\varepsilon} = \psi(a) e^{ika}$ whenever $a < 0 < b$. Taking the points a and b to approach the origin yields $\psi(0^-) = e^{-i\varepsilon} \psi(0^+)$, that is

$$s = e^{-i\varepsilon}. \quad (18)$$

B. When the phase jump will be fractional linear in the coupling parameter.

We now take the potential to have the form

$$V = |\delta_n\rangle\langle\delta_n|. \quad (19)$$

This leads to the stationary state equation

$$i\psi'(x) + \langle\delta_n|\psi\rangle\delta_n(x) = k\psi(x)$$

and one of the obvious features is that the function δ_n does not stay in the Hilbert space as $n \rightarrow \infty$. As an ansatz, we try a solution of the form

$$\psi_n(x) = \alpha_n \theta_n(x) + \phi(x) \quad (20)$$

where α_n is a complex scalar and $\theta_n(x) = \int_{-\infty}^x \delta_n(x') dx'$. The function ϕ is assumed to be continuous and differentiable at $x = 0$. Substituting the trial function, we find

$$i\alpha_n \delta_n(x) + i\phi'(x) + \varepsilon \langle \delta_n | \psi_n \rangle \delta_n(x) = k\alpha_n \theta_n(x) + k\phi(x),$$

and to remove the divergent $\delta_n(x)$ term we must take $\alpha_n = i\varepsilon \langle \delta_n | \psi_n \rangle$. This leaves $i\phi'(x) = k\alpha_n \theta_n(x) + k\phi(x)$ which integrates to

$$e^{ikb} \phi(b) - e^{ika} \phi(a) = ik\alpha_n \int_a^b e^{ikx} \theta_n(x) dx \quad (21)$$

and the right hand side converges to $\alpha(e^{ikb} - 1)$ as $n \rightarrow \infty$, whenever $a < x < b$. Again, taking a and b approaching zero we get the consistency condition $\phi(0^-) = \phi(0^+)$. The limit function ψ is then $\psi(x) = \alpha\theta(x) + \phi(x)$ where θ is the Heaviside function and $\alpha = \lim_{n \rightarrow \infty} \alpha_n = i\varepsilon \frac{\psi(0^+) + \psi(0^-)}{2}$. We then have $\psi(0^+) = \psi(0^-) + \alpha$. Eliminating α then leads to $\psi(0^-) = \frac{1 - \frac{i}{2}\varepsilon}{1 + \frac{i}{2}\varepsilon} \psi(0^+)$, that is

$$s = \frac{1 - \frac{i}{2}\varepsilon}{1 + \frac{i}{2}\varepsilon}. \quad (22)$$

C. Remarks

The choice of $-p$ as free Hamiltonian lead to scattering coefficient s that is independent of the wavenumber k . We encounter two very different forms: the exponential $s_1(\varepsilon) = e^{-i\varepsilon}$ and the fractional linear $s_2(\varepsilon) = \frac{1 - \frac{i}{2}\varepsilon}{1 + \frac{i}{2}\varepsilon}$. Remarkably they agree up to second order when Taylor expanded in the coupling strength ε . This means that intuitive arguments based on perturbation expansions may not always be reliable.

III. STOCHASTIC JUMP EVOLUTIONS

We now second quantize the situation encountered in the Section II. We consider a single quantum input process $b(t)$ satisfying singular commutation relations $[b(t), b(s)^\dagger] = \delta(t - s)$. and set $\Lambda(t) = \int_0^t b^\dagger(s)b(s)ds$, also known as the gauge process. The relevant quantum Itô product rule is $(d\Lambda(t))^2 = d\Lambda(t)$, and we have the Itô formula $df(\Lambda_t) = \{f(\Lambda(t) + 1) - f(\Lambda(t))\}d\Lambda(t)$. The general form of a pure-gauge unitary evolution coupling the system to the input field takes the form [1]

$$U(t) = (S \otimes I)^{I \otimes \Lambda(t)} \quad (23)$$

satisfying the quantum stochastic differential equation $dU(t) = (S - 1) \otimes d\Lambda(t) U(t)$, where S is required to

be a unitary operator on the system space. This is a degenerate triple $(S, 0, 0)$, that is the coupling parameters $L = 0$ and the Hamiltonian $H = 0$ in (4).

We now show the quantum stochastic analogues to the two types of limit encountered in the previous section. For convenience, we restrict to a single input field, but the generalisation to multiple modes is straightforward.

A. When the scattering matrix will be an exponential of the exchange matrix.

Fix a self-adjoint operator E_{11} on the system Hilbert space, then chose the time-dependent Hamiltonian

$$\tilde{H}^{(n)}(t) = E_{11} \otimes \tilde{\lambda}^{(n)}(t), \quad (24)$$

with $\tilde{\lambda}^{(n)}(t) = \int \delta_n(t - s) d\Lambda(s) \equiv \int \delta_n(t - s) b^\dagger(s)b(s)ds$. We denote by $\tilde{U}^{(n)}(t)$ the solution to the corresponding Schrödinger equation

$$i \frac{d}{dt} \tilde{U}^{(n)}(t) = \tilde{H}^{(n)}(t) \tilde{U}^{(n)}(t) \quad (25)$$

with $\tilde{U}^{(n)}(0) = I$. Here the solution will be $\tilde{U}^{(n)}(t) = \exp\{-iE_{11} \otimes \int_0^t ds \int \delta_n(s - u) d\Lambda(u)\}$, but

$$\int_0^t ds \int \delta_n(s - u) d\Lambda(u) = \int \delta_n * 1_{[0,t]}(u) d\Lambda(u) \quad (26)$$

and we encounter the convolution $\delta_n * 1_{[0,t]}$ of the approximate delta function δ_n with the indicator function $1_{[0,t]}$ of the interval $[0, t]$. We then have the strongly convergent limit to $\Lambda(t)$ and so $\tilde{U}^{(n)}(t)$ is strongly convergent to $\tilde{U}(t) = \exp\{-iE_{11} \otimes \Lambda(t)\}$. This of course corresponds to the pure gauge driven unitary with scattering matrix $\tilde{S} = e^{-iE_{11}}$.

In the multiple input field case we have the matrix relation

$$\tilde{S} = e^{-iE_{\ell\ell}} \quad (27)$$

N.B. Recall that the entries E_{jk} of the exchange matrix are operators on the system.

This limit is naturally associated with the Hølevø time-ordered exponential form of the quantum stochastic calculus as, indeed,

$$\tilde{U}(t) = \vec{T}_H e^{-i \sum_{jk} E_{jk} \Lambda_{jk}(t)}. \quad (28)$$

B. When the scattering matrix will be fractional linear in the exchange matrix.

We alternatively take

$$H^{(n)}(t) = E_{11} \otimes \tilde{b}^{(n)}(t)^\dagger \tilde{b}^{(n)}(t) \quad (29)$$

where $\tilde{b}^{(n)}(t) = \int \delta_n(t - s) b(s) ds$ is a smeared annihilator, etc. The unitaries $U_n(t)$ generated by time-dependent Hamiltonian $H^{(n)}(t)$ converge to the unitary quantum

stochastic process $U(t)$ with triple $(S, 0, 0)$ where $S = \frac{1 - \frac{i}{2}E_{11}}{1 + \frac{i}{2}E_{11}}$, [12].

In the multiple input field case we then have the matrix relation

$$S = \frac{1 - \frac{i}{2}E_{\ell\ell}}{1 + \frac{i}{2}E_{\ell\ell}} \quad (30)$$

This limit is naturally associated with the Stratonovich form as now

$$U(t) = \bar{T}_D e^{-i \sum_{jk} E_{jk} \Lambda_{jk}(t)}. \quad (31)$$

C. Adiabatic Elimination of a Cavity Mode

In [2] the adiabatic elimination of a cavity mode b was considered where the mode had a Hamiltonian of the form

$$H^{(n)} = E_{00} + \sqrt{n}E_{10}b^* + \sqrt{n}E_{01}b + nE_{11}b^\dagger b$$

with the mode coupled to an external field with a coupling strength $\sqrt{n}\gamma$. That is, we have the QSDE

$$dU_t = \left\{ \sqrt{n}\gamma b \otimes dB_t^* - \sqrt{n}\gamma b^* \otimes dB_t - \frac{n\gamma}{2} b^* b dt + iH^{(n)} dt \right\} U_t.$$

If we now introduce the unperturbed dynamics $dV_t = \{\sqrt{n}\gamma b \otimes dB_t^* - \sqrt{n}\gamma b^* \otimes dB_t - \frac{n\gamma}{2} b^* b dt\} V_t$, then it is shown that the unitary $\tilde{U}_t = V_t^{-1} U_t$ satisfies a limit QSDE as $n \rightarrow \infty$ of the form (3) with (S, L, H) given by

$$\begin{aligned} S &= \frac{\gamma/2 - iE_{11}}{\gamma/2 + iE_{11}}, \\ L &= \frac{i\sqrt{\gamma}}{\gamma/2 + iE_{11}}, \\ H &= E_{00} + E_{01} \text{Im} \left\{ \frac{1}{\gamma/2 + iE_{11}} \right\} E_{01}. \end{aligned}$$

In the special case of an atomic system in a cavity, one may consider [14]

$$E_{11} = -\frac{g_0^2}{\Delta} \cos^2(kq), \quad E_{01} = 0 = E_{10}, \quad E_{00} = H$$

and so

$$\begin{aligned} S &= \frac{\gamma/2 + i\frac{g_0^2}{\Delta} \cos^2(kq)}{\gamma/2 - i\frac{g_0^2}{\Delta} \cos^2(kq)} \\ &= \exp \left(2i \tan^{-1} \left(\frac{2g_0^2}{\gamma\Delta} \cos^2(kq) \right) \right). \end{aligned} \quad (32)$$

In the limit model, we find that the atomic indeed induces a phase change on the optical field.

IV. VARIABLE-SPEED QUANTUM TRAVELING FIELD MODES

We now give a non-perturbative argument leading to jump QSDEs starting from scattering of light by quantum systems which correspond to free boundaries. Our approach is to use the theory developed by Ley and Loudon [15], see also [16], where the quantise the classical mode fields for electromagnetic fields scattered by dielectric media, as opposed to trying to begin with free photons. We mention other approaches such as [17] which deal with quantisation of light in dielectric material.

We begin with Maxwell's equations without sources

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, \quad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}, \\ \nabla \cdot \mathbf{D} &= 0, \quad \nabla \times \mathbf{H} = \frac{\partial}{\partial t} \mathbf{D}. \end{aligned}$$

Our interest is in the situation where the displacement field takes the form $\mathbf{D} = \varepsilon \mathbf{E}$ with a dielectric coefficient ε which depends on position. We shall take $\mathbf{B} = \mu_0 \mathbf{H}$ with constant permeability μ_0 . The first pair of equations lead to the usual potential

$$\mathbf{E} = -\frac{\partial}{\partial t} \mathbf{A} - \nabla \phi, \quad \mathbf{B} = \nabla \times \mathbf{A},$$

and we will fix the Coulomb gauge ($\phi \equiv 0$).

We seek to model a field propagating in a thin wire along the z -axis of cross-section \mathcal{A} , and to this end we take the vector potential to have non-zero component $A_x = A_x(z, t)$ in which case the non-zero components of the electric and magnetic fields are

$$E_x(z, t) = -\frac{\partial}{\partial t} A_x(z, t), \quad B_y(z, t) = \frac{\partial}{\partial z} A_x(z, t).$$

We also take the dielectric constant to be a function of the z coordinate only and set

$$\zeta(z) = \varepsilon(z) \mu_0 \equiv \frac{1}{c^2} n(z)^2 \quad (33)$$

where $n(z)$ is a position dependent refractive index. We shall assume the asymptotic behaviour

$$\lim_{z \rightarrow \infty} \zeta(z) = \left(\frac{n_r}{c} \right)^2, \quad \lim_{z \rightarrow -\infty} \zeta(z) = \left(\frac{n_l}{c} \right)^2.$$

Here $n_r \geq 1$ and $n_l \geq 1$ are the refractive indices in the right and left far zones respectively. The equation $\nabla \cdot \mathbf{D} = 0$ is now trivially satisfied, and with the remaining Maxwell's equation we see that the component A_x satisfies the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \zeta(z) \frac{\partial^2}{\partial t^2} \right) A_x(z, t) = 0. \quad (34)$$

A. Mode Functions

Let $\omega > 0$ be a positive frequency, and consider trial solutions to (34) of the form $U_\omega(z) e^{-i\omega t}$. We see that the mode functions U_z satisfy

$$\left(\frac{d^2}{dz^2} + \zeta(z) \omega^2 \right) U_\omega(z) = 0. \quad (35)$$

The mode corresponding to a right incoming traveling wave is the solution $U_{\omega,r}$ with the asymptotic behavior

$$U_{\omega,r}(z) \simeq \begin{cases} e^{-in_r \omega z/c} + t_{rr}(\omega) e^{in_r \omega z/c}, & z \rightarrow \infty; \\ t_{lr}(\omega) e^{-in_l \omega z/c}, & z \rightarrow -\infty. \end{cases}$$

We may similarly introduce the left incoming mode as the solution with the asymptotic behavior

$$U_{\omega,l}(z) \simeq \begin{cases} t_{rl}(\omega) e^{in_r \omega z/c}, & z \rightarrow \infty; \\ e^{in_l \omega z/c} + t_{ll}(\omega) e^{-in_l \omega z/c}, & z \rightarrow -\infty. \end{cases}$$

See Fig. 3.

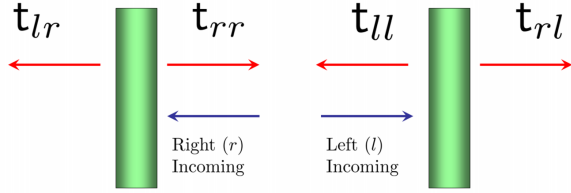


FIG. 3. (Color online) In the far zone: the right incoming plane wave has reflected coefficient t_{rr} and transmitted coefficient t_{lr} ; likewise the left incoming plane wave has reflected coefficient t_{ll} and transmitted coefficient t_{rl} .

The fields may be decomposed as $A(z, t) = A^+(z, t) + A^-(z, t)$ where positive-frequency components are then given by

$$A^+(z, t) = \int_0^\infty \sqrt{\frac{\hbar}{2\pi\mathcal{A}\omega}} \sum_{c=r,l} U_{\omega,c}(z) a_c(\omega) e^{-i\omega t} d\omega$$

$$E^+(z, t) = i \int_0^\infty \sqrt{\frac{\hbar\omega}{2\pi\mathcal{A}}} \sum_{c=r,l} U_{\omega,c}(z) a_c(\omega) e^{-i\omega t} d\omega$$

$$B^+(z, t) = \int_0^\infty \sqrt{\frac{\hbar}{2\pi\mathcal{A}\omega}} \sum_{c=r,l} U'_{\omega,c}(z) a_c(\omega) e^{-i\omega t} d\omega$$

while the negative-frequency fields are just the hermitean conjugates $A^-(z, t) = A^+(z, t)^\dagger$, etc. The field is quantized by introducing the canonical commutation relations

$$[a_b(\omega), a_c^\dagger(\omega')] = \delta_{bc} \delta(\omega - \omega'), \quad (36)$$

where $b, c \in \{r, l\}$.

B. Far Field Input/Output Relations

The electromagnetic field has the (Abrahams) momentum density is $\mathbf{g} = \frac{1}{c^2} \mathbf{E} \times \mathbf{H}$ which in the present case has non-zero component $g_z = \frac{1}{\varepsilon_0} E_x B_y$. Taking a *classical* potential $A^+(z, t) = \sum_{c=r,l} U_{\omega,c}(z) \alpha_c e^{-i\omega t}$ for a pair of complex constants α_r, α_l . The far right and far left one-cycle time-averaged values should be equal in order to have conservation of electromagnetic momentum and from this we derived the following identities

$$\begin{aligned} n_r |t_{rr}(\omega)|^2 + n_l |t_{lr}(\omega)|^2 &= n_r, \\ n_r |t_{rl}(\omega)|^2 + n_l |t_{ll}(\omega)|^2 &= n_l, \\ n_r t_{rr}(\omega)^* t_{rl}(\omega) + n_l t_{lr}(\omega)^* t_{ll}(\omega) &= 0, \end{aligned}$$

due to the arbitrariness of α_r and α_l . A little algebra also reveals that we have $n_l^2 |t_{lr}(\omega)|^2 = n_r^2 |t_{rl}(\omega)|^2$. The matrix $\mathbf{t}(\omega) = \begin{bmatrix} t_{rr}(\omega) & t_{rl}(\omega) \\ t_{lr}(\omega) & t_{ll}(\omega) \end{bmatrix}$ will be unitary in the special case where the left and right refractive indices are equal ($n_l = n_r$).

In general however the relations between the left and right far field components can be expressed by saying that the matrix

$$S(\omega) = \begin{bmatrix} t_{rr}(\omega) & \sqrt{\frac{n_r}{n_l}} t_{rl}(\omega) \\ \sqrt{\frac{n_l}{n_r}} t_{lr}(\omega) & t_{ll}(\omega) \end{bmatrix} \quad (37)$$

is unitary.

C. Orthogonality of the Modes

We now generalize the results of Ley and Loudon [15] to $n_r, n_l \neq 1$. From the observation that

$$(U_{\omega_1}^* U'_{\omega_2} - U_{\omega_1}' U_{\omega_2})' = (\omega_1^2 - \omega_2^2) \zeta U_{\omega_1}^* U_{\omega_2}$$

we see that

$$\begin{aligned} & \int_{-M}^L U_{\omega_1}^*(z) U_{\omega_2}(z) \zeta(z) dz \\ &= \frac{1}{(\omega_1^2 - \omega_2^2)} (U_{\omega_1}^* U'_{\omega_2} - U_{\omega_1}' U_{\omega_2}) \Big|_{z=-M}^L. \end{aligned}$$

For the right incoming mode we have, for large M and L ,

$$\begin{aligned} & \frac{1}{(\omega_1^2 - \omega_2^2)} \left\{ e^{in_r(\omega_1 - \omega_2)L/c} \frac{(\omega_1 + \omega_2) n_r}{ic} \right. \\ & + e^{-in_r(\omega_1 - \omega_2)L/c} \frac{(\omega_1 + \omega_2) n_r}{ic} t_{rr}(\omega_1)^* t_{rr}(\omega_2) \\ & \left. + e^{-in_l(\omega_1 - \omega_2)M/c} \frac{(\omega_1 + \omega_2) n_l}{ic} t_{lr}(\omega_1)^* t_{lr}(\omega_2) + \dots \right\}. \end{aligned}$$

The remaining terms, appearing as an ellipsis, are proportional to $(\omega_1 - \omega_2) \times e^{\pm i(n_r \omega_1 L + n_l \omega_2 M)/c}$ and will not

contribute. We need to take $M = \frac{n_r}{n_l}L$ to obtain a balanced limit which leads us to define the following skewed principal value integral:

$$\oint_{-\infty}^{\infty} f(z) dz := \lim_{N \rightarrow \infty} \int_{-N/n_l}^{N/n_r} f(z) dz$$

in which case

$$\begin{aligned} & \oint_{-\infty}^{\infty} U_{\omega_1, r}^*(z) U_{\omega_2, r}(z) \zeta(z) dz \\ &= \lim_{N \rightarrow \infty} 2N \operatorname{sinc}(N(\omega_1 - \omega_2)) \\ &= 2\pi \delta(\omega_1 - \omega_2), \end{aligned}$$

where $\operatorname{sinc}(x) = \frac{\sin x}{x}$. Here we use the flux relations with $\omega \rightarrow 0$. The first three terms contribute while the remainder contributes an unsupported $\delta(\omega_1 + \omega_2)$ which is ignored. (Note that the sinc function is an improper integral and not absolutely integrable, therefore the choice of skew adopted is necessary to obtain the nascent delta function limit.)

One finds that the various modes are orthogonal in the sense that

$$\int_{-\infty}^{\infty} U_{\omega_1, b}^*(z) U_{\omega_2, c}(z) \zeta(z) dz = 2\pi \delta_{bc} \delta(\omega_1 - \omega_2).$$

The sesquilinear form appearing here is the correct notion to formulate the Sturm-Liouville orthogonality property for the modes.

D. The Hamiltonian

The Hamiltonian is then taken to be

$$\begin{aligned} H &= \frac{1}{2} \oint \left[\varepsilon(z) E^- E^+(z, t) + \frac{1}{\mu_0} B^- B^+(z, t) \right] dV \\ &= \int_0^{\infty} \hbar \omega \left[a_r^\dagger(\omega) a_r(\omega) + a_l^\dagger(\omega) a_l(\omega) \right] d\omega, \end{aligned} \quad (38)$$

where the z -integration is interpreted as the skewed principal value.

V. TWO-SIDED MODELS

In this section we present several situations of interest.

A. Left and Right regions with unequal constant dielectric coefficient

We have a constant dielectric coefficients in two semi-infinite regions with boundary at $z = q$:

$$\zeta(z) = \begin{cases} \left(\frac{n_r}{c}\right)^2, & z > q; \\ \left(\frac{n_l}{c}\right)^2, & z < q. \end{cases}$$

The modes will be piecewise plane waves, for instance,

$$U_{\omega, r}(z) = \begin{cases} e^{-in_r \omega z/c} + \mathbf{t}_{rr}(\omega) e^{in_r \omega z/c}, & z > q; \\ \mathbf{t}_{lr}(\omega) e^{-in_r \omega z/c}, & z < q. \end{cases}$$

Continuity of $U_{\omega, r}$ and $U'_{\omega, r}$ across the boundary yields

$$\begin{aligned} \mathbf{t}_{rr}(\omega) &= -\frac{n_l - n_r}{n_l + n_r} e^{-2in_r \omega q/c}, \\ \mathbf{t}_{lr}(\omega) &= \frac{2n_r}{n_l + n_r} e^{-i(n_r - n_l) \omega q/c}. \end{aligned}$$

and one similarly calculates that $\mathbf{t}_{ll}(\omega) = \frac{n_l - n_r}{n_l + n_r} e^{2in_l \omega q/c}$, $\mathbf{t}_{rl}(\omega) = \frac{2n_l}{n_l + n_r} e^{-i(n_r - n_l) \omega q/c}$. The matrix $S(\omega)$ is then given by

$$\begin{bmatrix} \frac{(n_r - n_l)}{n_r + n_l} e^{-2in_r \omega q/c} & \frac{2\sqrt{n_r n_l}}{n_r + n_l} e^{-i(n_r - n_l) \omega q/c} \\ \frac{2\sqrt{n_r n_l}}{n_r + n_l} e^{-i(n_r - n_l) \omega q/c} & \frac{(n_l - n_r)}{n_r + n_l} e^{2in_l \omega q/c} \end{bmatrix}. \quad (39)$$

B. Dielectric Slab

A dielectric slab of thickness $2a$ about $z = q$ is modeled by

$$c^2 \zeta(z) = \begin{cases} n_r^2, & I : z > q + a; \\ n_l^2, & II : |z - q| < a; \\ n_l^2, & III : z < q - a. \end{cases}$$

We now set

$$U_{\omega, r}(z) = \begin{cases} e^{-in_r \omega z/c} + \mathbf{t}_{rr}(\omega) e^{in_r \omega z/c}, & I; \\ A(\omega) e^{-in_r \omega z/c} + B(\omega) e^{in_r \omega z/c}, & II; \\ \mathbf{t}_{lr}(\omega) e^{-in_r \omega z/c}, & III. \end{cases}$$

Again requiring continuity of $U_{\omega, r}$ and $U'_{\omega, r}$ across the boundaries leads to the expressions for $\mathbf{t}_{rr}(\omega)$ and $\mathbf{t}_{lr}(\omega)$:

$$\begin{aligned} \mathbf{t}_{rr}(\omega) &= \frac{1}{D(\omega)} \{ (n + n_l)(n - n_r) e^{2ina\omega/c} \\ &\quad - (n - n_l)(n + n_r) e^{-2ina\omega/c} \} e^{-2in_r(q+a)\omega/c}, \\ \mathbf{t}_{lr}(\omega) &= -\frac{1}{D(\omega)} 4nn_r e^{i(n_l - n_r)q\omega/c} e^{-i(n_r + n_l)a\omega/c}, \end{aligned}$$

with the denominator $D(\omega)$

$$(n - n_r)(n - n_l) e^{2ina\omega/c} - (n + n_r)(n + n_l) e^{-2ina\omega/c}.$$

The left incoming coefficients are obtained by symmetry: $\mathbf{t}_{rr} \leftrightarrow \mathbf{t}_{ll}$ and $\mathbf{t}_{rl} \leftrightarrow \mathbf{t}_{lr}$ under the parameter inversion $a \leftrightarrow -a$, $n_r \leftrightarrow -n_l$ and $n_l \leftrightarrow -n_r$.

C. Singular Dielectric Boundaries

It is of interest to consider the limit of vanishing thickness with large internal refractive index. Specifically we consider the previous model of a dielectric slab and take the limits

$$a \rightarrow 0 \text{ with } 2n^2 a = \mu \text{ (constant).}$$

The limiting forms are then

$$\begin{aligned} t_{rr}(\omega) &= \frac{(n_r - n_l) + i\mu\omega/c}{(n_r + n_l) - i\mu\omega/c} e^{-2in_r\omega q/c}, \\ t_{lr}(\omega) &= \frac{2n_r}{(n_r + n_l) - i\mu\omega/c} e^{i(n_l - n_r)\omega q/c}. \end{aligned}$$

The left incoming coefficients are similarly calculated (the parameter inversion is now $\mu \leftrightarrow -\mu$, $n_r \leftrightarrow -n_l$ and $n_l \leftrightarrow -n_r$) and one has

$$S(\omega) = \frac{1}{(n_r + n_l) - i\mu\omega/c} \times \begin{bmatrix} \left(n_r - n_l + i\frac{\mu\omega}{c} \right) e^{-2in_r\frac{\omega}{c}q} & 2\sqrt{n_r n_l} e^{i(n_l - n_r)\frac{\omega}{c}q} \\ 2\sqrt{n_r n_l} e^{i(n_l - n_r)\frac{\omega}{c}q} & \left(n_l - n_r + i\frac{\mu\omega}{c} \right) e^{2in_l\frac{\omega}{c}q} \end{bmatrix}$$

D. Singular Dielectric Points

In particular, if we set $n_l = n_r = 1$, then we are left with the singular dielectric point at $z = q$:

$$S(\omega) = \frac{1}{2 - i\mu\omega/c} \begin{bmatrix} \frac{i\mu\omega}{c} e^{-2i\omega q/c} & 2 \\ 2 & -\frac{i\mu\omega}{c} e^{2i\omega q/c} \end{bmatrix} \quad (40)$$

This may be formally understood as arising from the singular distribution

$$\zeta(z) = 1 + \mu\delta(z - q).$$

To see this, we take the general solution for $U_{\omega,r}$,

$$U_{\omega,r}(z) = \begin{cases} e^{-i\omega z/c} + t_{rr}(\omega) e^{i\omega z/c}, & z > q; \\ t_{lr}(\omega) e^{-i\omega z/c}, & z < q; \end{cases}$$

and impose continuity of $U_{\omega,r}$ at the boundary q , along with the condition

$$U'_{\omega,r}(q^-) - U'_{\omega,r}(q^+) = \mu \frac{\omega^2}{c^2} U_{\omega,r}(q). \quad (41)$$

Physically this is interpreted as a discontinuity in B_y due the finite change in the time-derivative of the displacement D_x across the infinitesimal boundary [15]. A similar condition applies to $U_{\omega,l}$.

VI. MIRRORS

Mirrors are special cases where the light comes exclusively from one direction (the right say) and is reflected back. This leads to a semi-infinite geometry.

A. Perfect Mirrors

Perfect Mirrors can be viewed as the limiting situation where, say, the left refractive index n_l becomes infinite. For instance, taking a boundary at $z = q$, we ignore the left-incoming wave and find that the right incoming wave is reflected with unimodular coefficient

$$r_r(\omega) = \lim_{n_l \rightarrow \infty} t_{rr}(\omega) = -e^{-2in_r\omega q/c}, \quad (42)$$

where we take the $n_l \rightarrow \infty$ limit of (39). Here the phase includes the information of the boundary position q .

B. Singular Dielectric Boundary

An alternative model would be to have a fixed perfect mirror at $z = 0$ and a singular dielectric boundary at $z = q$. In this case we have $z \geq 0$ only and set

$$U_{\omega,r}(z) = \begin{cases} e^{-i\omega z/c} + r_r(\omega) e^{i\omega z/c}, & z > q; \\ A(\omega) e^{-i\omega z/c} + B(\omega) e^{i\omega z/c}, & 0 \leq z < q. \end{cases}$$

The boundary conditions are $U_{\omega,r}(0) = 0$ (so $A(\omega) = -B(\omega)$), $U_{\omega,r}(q^-) = U_{\omega,r}(q^+)$, and (41). One finds that $r_r(\omega)$ is again unimodular and given by

$$r_r(\omega) = \frac{1 - i\frac{\mu\omega}{2c} (e^{-2i\omega q/c} - 1)}{1 + i\frac{\mu\omega}{c} (e^{2i\omega q/c} - 1)}. \quad (43)$$

We note that $r_r(\omega) \rightarrow -e^{-2i\omega q/c}$ as the strength μ of the infinitesimally thin dielectric becomes infinite: that is we recover the limit of a perfect mirror at $z = q$.

C. Dielectric Layer

A similar situation is to have a perfect mirror at $z = 0$ and a dielectric slab between 0 and $z = q$ with fixed refractive index n , see Fig. 4. It is not too difficult to show that the reflection is now given by

$$r_r(\omega) = -e^{-2i\omega q/c} \frac{\cos\left(\frac{n\omega q}{c}\right) + \frac{i}{n} \sin\left(\frac{n\omega q}{c}\right)}{\cos\left(\frac{n\omega q}{c}\right) - \frac{i}{n} \sin\left(\frac{n\omega q}{c}\right)}.$$

VII. INPUT-OUTPUT FORMULATION

We now wish to introduce an input-output formalism based on measurements by detectors at positions Z_R and Z_L positioned in the right and left far zones of the field respectively. To avoid an number of issues involved with the one dimensional nature of the fields, we shall assume that both far zones are non-dielectric, that is

$$n_r = n_l = 1.$$

Note that we now have

$$S(\omega) \equiv \begin{bmatrix} t_{rr}(\omega) & t_{rl}(\omega) \\ t_{lr}(\omega) & t_{ll}(\omega) \end{bmatrix} \quad (44)$$

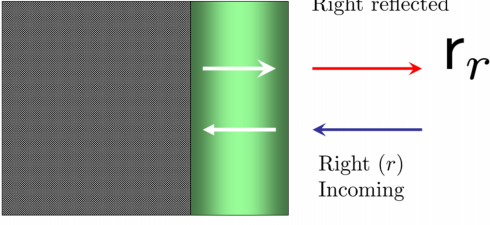


FIG. 4. (Color online) An imperfect mirror may be modelled as a perfect mirror with a dielectric layer of refractive index n .

which will be unitary.

The electric field at the right detector is given approximately by

$$E^+(Z_R, t) \simeq i \int_0^\infty \sqrt{\frac{\hbar\omega}{2\pi\mathcal{A}}} e^{-i\omega(t+Z_R/c)} a_r(\omega) d\omega \\ + i \int_0^\infty \sqrt{\frac{\hbar\omega}{2\pi\mathcal{A}}} \mathbf{t}_{rr}(\omega) e^{-i\omega(t-Z_R/c)} a_r(\omega) d\omega \\ + i \int_0^\infty \sqrt{\frac{\hbar\omega}{2\pi\mathcal{A}}} \mathbf{t}_{rl}(\omega) e^{-i\omega(t-Z_R/c)} a_l(\omega) d\omega.$$

We now make the standard quantum white noise assumption for optical fields: this amounts to identifying a central frequency Ω and replace the $\sqrt{\omega}$ and $\mathbf{t}_{ab}(\omega)$ terms with their values at $\omega = \Omega$, and otherwise taking the lower value of the integral to $-\infty$:

$$E^+(Z_R, t) \simeq i \sqrt{\frac{\hbar\Omega}{2\pi\mathcal{A}}} b_r(t + Z_R/c) \\ + i \sqrt{\frac{\hbar\Omega}{2\pi\mathcal{A}}} \mathbf{t}_{rr}(\Omega) b_r(t - Z_R/c) \\ + i \sqrt{\frac{\hbar\Omega}{2\pi\mathcal{A}}} \mathbf{t}_{rl}(\Omega) b_l(t - Z_R/c).$$

where $b_k(\tau) = \int_{-\infty}^\infty e^{-i\omega\tau} a_k(\omega) d\omega$ for $k = r, l$.

Measurement of the electric field may then effectively be a measurement (e.g., homodyne quadrature, photon counting, etc.) of the scattered output field

$$b_r^{\text{out}}(t) = \mathbf{t}_{rr}(\Omega) b_r(t - Z_R/c) + \mathbf{t}_{rl}(\Omega) b_l(t - Z_R/c)$$

with a similar expression for the left field. Ignoring the time delays, we may write the input-output equations as

$$\begin{bmatrix} b_r^{\text{out}}(t) \\ b_l^{\text{in}}(t) \end{bmatrix} = \begin{bmatrix} S_{rr} & S_{rl} \\ S_{lr} & S_{ll} \end{bmatrix} \begin{bmatrix} b_r(t) \\ b_l(t) \end{bmatrix}.$$

The coefficients S_{ab} are the transmission/reflection coefficients $\mathbf{t}_{ab}(\Omega)$ and it is convenient to introduce the photon momentum

$$k = \Omega/c.$$

We also have that these coefficients depend on the details of (one or more) free boundaries q . We now lift the condition that the q need to be fixed parameters and allow them to be quantum mechanical. In particular, they are time varying also.

A. QSDE Model

We now introduce a quantum stochastic differential equation (QSDE) that leads to the above input-output models. We consider the QSDE corresponding to vacuum inputs:

$$\dot{U}_t = b_a(t)^* S_{ab} U_t b_b(t) - i H U_t$$

with initial condition $U_0 = 1$. (Here repeated indices imply a sum over the values r and l .) The Hamiltonian term is taken to have a standard form

$$H = \frac{1}{2m} p^2 + V(q).$$

Let us define the processes

$$B_a(t) = \int_0^t b_a(t') dt', \quad \Lambda_{ab}(t) = \int_0^t b_a(t')^* b_b(t') dt'$$

then the QSDE may be written alternatively as

$$dU_t = \{(S_{ab} - \delta_{ab}) d\Lambda_{ab} - i H dt\} U_t.$$

The coefficients S_{ab} are now taken to depend on the position operator q of the free quantum boundary. They make up a scattering matrix S which is therefore a two-by-two matrix with position-operator dependent entries, and S is unitary.

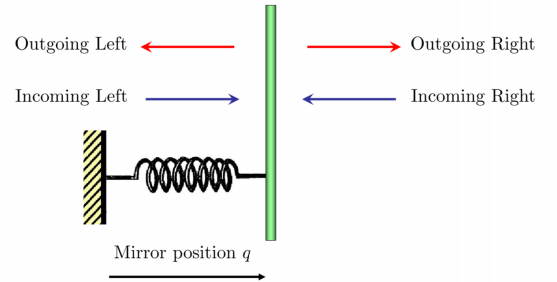


FIG. 5. (Color online) Dielectric “particle” scattered by light.

Given an arbitrary observable X of the system, its value at time t will be

$$X_t = U_t^* (X \otimes I) U_t$$

and from the quantum Itô calculus we find

$$\dot{X}_t = b_a^*(t) (\mathcal{L}_{ab} X)_t b_t - i [X, H]_t$$

where

$$\mathcal{L}_{ab} X = S_{ca}^* X S_{cb} - \delta_{ab} X.$$

Now as the S_{ab} are functions of the position observable q only we have

$$\begin{aligned}\mathcal{L}_{ab}q &= 0, \\ \mathcal{L}_{ab}p &= -i\hbar S_{ca}^* S'_{cb}\end{aligned}$$

where the prime denotes differentiation with respect to the *variable* q . We therefore obtain the position and momentum QSDEs

$$\begin{aligned}\dot{q}_t &= \frac{1}{m}p_t, \\ \dot{p}_t &= -V'(q_t) - i\hbar b_a^*(t) (S_{ca}^* S'_{cb}) b_b(t).\end{aligned}\quad (45)$$

The vacuum average yields the usual Ehrenfest equations, however, to obtain something nontrivial we consider a coherent state input field with intensity $\beta_a(t)$, for $a = r, l$. This is equivalent to making the replacement $b_a(t) \rightarrow b_a(t) + \beta_a(t)$ so that the QSDE becomes

$$\dot{U}_t = (b_a(t) + \beta_a(t))^* S_{ab} U_t (b_b(t) + \beta_b(t)),$$

or,

$$\begin{aligned}dU_t &= \{(S_{ab} - \delta_{ab}) d\Lambda_{ab} + (S_{ab} - \delta_{ab}) \beta_a^* dB_b \\ &\quad + (S_{ab} - \delta_{ab}) \beta_b dB_a^* + (S_{ab} - \delta_{ab}) \beta_a^* \beta_b dt\} U_t \\ &\quad - iH U_t dt.\end{aligned}\quad (46)$$

We now obtain the position and momentum QSDEs

$$\begin{aligned}\dot{q}_t &= \frac{1}{m}p_t, \\ \dot{p}_t &= -V'(q_t) \\ &\quad - i\hbar (b_a^*(t) + \beta_a^*(t)) (S_{ca}^* S'_{cb}) (b_b(t) + \beta_b(t)).\end{aligned}$$

The averages now lead to

$$\begin{aligned}\frac{d}{dt} \langle q_t \rangle &= \frac{1}{m} \langle p_t \rangle, \\ \frac{d}{dt} \langle p_t \rangle &= -\langle V'(q_t) \rangle - i\hbar \beta_a^*(t) (S_{ca}^* S'_{cb}) \beta_b(t).\end{aligned}$$

In the special case of a mirror, where we have only an input $b = b_r$ from the right (say) then the equation (45) above simplifies to

$$\begin{aligned}\dot{q}_t &= \frac{1}{m}p_t, \\ \dot{p}_t &= -V'(q_t) + \hbar b^*(t) \theta'(q_t) b(t).\end{aligned}\quad (47)$$

where β is the intensity of the coherent input field from the right, and

$$S = e^{i\theta(q)}$$

is the reflection coefficient.

VIII. EXAMPLES

A. Mirrors

In the case of a perfect mirror at position q we have form (42) $R = e^{-i2kq}$ and so (47) yields

$$\dot{p}_t = -V'(q_t) - 2\hbar k b^*(t) b(t).$$

This has the natural interpretation that the forcing term is the mechanical force due to the potential V and the radiation pressure which is the de Broglie momentum $\hbar k$ of the photon (doubled due to the reflection) times the number intensity $b^*(t) b(t)$. In a coherent state of intensity β , this yields

$$\frac{d}{dt} \langle p_t \rangle = -\langle V'(q_t) \rangle - 2\hbar k |\beta(t)|^2.$$

In the case of a singular dielectric boundary at q we obtain from (43)

$$\begin{aligned}\dot{p}_t &= -V'(q_t) \\ &\quad - 2\hbar k b^*(t) \frac{\mu^2 k^2 (\cos 2kq_t - 1) + 2\mu k \sin(2kq_t)}{\mu^2 k^2 (\cos 2kq_t - 1) - 2\mu k (\sin 2kq_t - 1)} b(t),\end{aligned}$$

which reduces to the perfect mirror expression in the limit $\mu \rightarrow \infty$.

B. A Dielectric Particle

We may consider a point particle of mass m with dielectric strength μ . The scattering matrix for photons of wave vector k is therefore from (40)

$$S = \frac{1}{1 - \frac{i}{2}\mu k} \begin{bmatrix} \frac{i}{2}\mu k e^{-2ikq} & 1 \\ 1 & -\frac{i}{2}\mu k e^{2ikq} \end{bmatrix}$$

and we have

$$S^* S' = \frac{k}{1 + \frac{1}{4}\mu^2 k^2} \begin{bmatrix} -\frac{i}{2}\mu^2 k^2 & \mu k e^{2ikq} \\ \mu k e^{-2ikq} & \frac{i}{2}\mu^2 k^2 \end{bmatrix}.$$

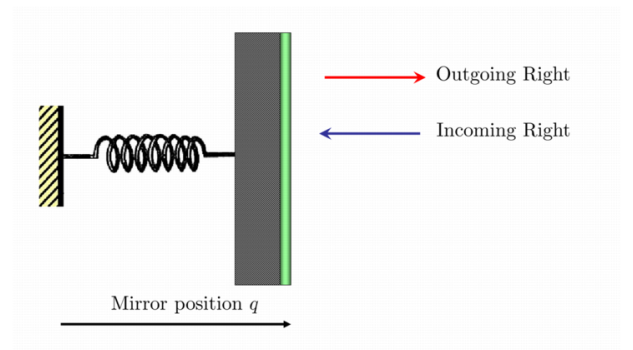


FIG. 6. (Color online) Optomechanical model where an imperfect mirror with quantum mechanical position q is in interaction with an input field.

The Langevin equation for p_t is in this case

$$\begin{aligned} \frac{d}{dt}p_t &= -V'(q_t) \\ &\quad - \frac{\hbar k}{1 + \frac{1}{4}\mu^2 k^2} \left\{ \frac{1}{2}\mu^2 k^2 [b_r^*(t)b_r(t) - b_l^*(t)b_l(t)] \right. \\ &\quad \left. + i\mu k b_r^*(t)e^{2ikq_t}b_l(t) + i\mu k b_l^*(t)e^{-2ikq_t}b_r(t) \right\}. \end{aligned}$$

In the limit $\mu \rightarrow \infty$ of infinite dielectric constant we obtain

$$\frac{d}{dt}p_t = -V'(q_t) - 2\hbar k [b_r^*(t)b_r(t) - b_l^*(t)b_l(t)]$$

consistent with a perfect two-sided mirror at q_t with left and right field quanta reflected with momenta $\pm\hbar k$ respectively.

C. The Adiabatic Elimination Example

We have the reflection coefficient $R = S$ given by (32) so that

$$\begin{aligned} \frac{d}{dt}p_t &= -V'(q_t) \\ &\quad - 4\hbar k b^*(t) \frac{\gamma \Delta g_0^2 \cos(2kq_t)}{\gamma^2 \Delta^2 + 4g_0^4 \cos^4(kq_t)} b(t). \end{aligned}$$

IX. QUANTUM MEASUREMENT

We now turn to the filtering problem, namely how do we best estimate the state of the mirror from observations of the reflected output fields. We consider a detector located in the far zone where we measure some observable $Y(t)$ of the field at time t . The time to go from the mirror to the detector will be assumed negligible. The set of observables $\{Y(s) : 0 \leq s \leq t\}$ is assumed to be commutative, and our aim is to calculate the conditional density matrix ϱ_t based on these observations [18, 19].

The most flexible approach is to use the use the theory of quantum filtering. Let \mathfrak{Y}_t be the von Neumann algebra generated by $\{Y(s) : 0 \leq s \leq t\}$. The we aim to compute, for each system operator X the conditional expectation

$$\widehat{X}_t = \mathbb{E}[j_t(X) | \mathfrak{Y}_t]$$

of the Heisenberg picture value of the operator at time t onto the algebra generated by the measurements up to that time. We shall use established results to derive explicit dynamical equations for $\pi_t(X)$ and therefore, through the identification

$$\widehat{X}_t = \text{tr}[\varrho_t X],$$

for ϱ itself. We shall use the filtering equations derived in [20] for coherent state inputs. We recall that the SLH triple in this problem

$$S = \begin{bmatrix} S_{rr} & S_{rl} \\ S_{lr} & S_{ll} \end{bmatrix},$$

with the components dependent on the observable q , and L vanishing. The filtering problem in the presence of fields in coherent states with amplitudes $\beta_r(t)$ and $\beta_l(t)$ respectively, is then equivalent to the vacuum filtering problem with non-zero coupling

$$L^{\beta(t)} = S \begin{bmatrix} \beta_r(t) \\ \beta_l(t) \end{bmatrix}.$$

This, of course, is explicitly contained in the unitary QSDE (46). Note that $L^{\beta(t)\dagger} L^{\beta(t)} = \|\beta(t)\|^2$ where we have the norm $\|\beta(t)\|^2 = |\beta_r(t)|^2 + |\beta_l(t)|^2$.

A. Homodyne Measurement

Let $B_{\text{in},k}(t) = \int_0^t b_k(s) ds$ be the input annihilation process, then we might aim to measure the fields

$$Y_a(t) = U(t)^\dagger \left[1 \otimes (B_{\text{in},a}(t) + B_{\text{in},a}(t)^\dagger) \right] U(t)$$

which gives the output quadrature for $a = l, r$. It follows from the quantum Itô calculus that

$$dY_a(t) = \sum_{b=l,r} j_t(S_{ab}) [dB_{\text{in},b}(t) + \beta_b(t) dt] + \text{H.c.}$$

The process is a diffusion with $(dY_a)^2 = dt$.

The filter equation for homodyne measurement is then, from equation (20) of [20],

$$\begin{aligned} d\widehat{X}_t &= \widehat{\mathcal{L}}\widehat{X}_t dt \\ &\quad + \sum_a \left\{ \sum_b \left(\widehat{XS_{ab}} \right)_t \beta_b(t) + \sum_b \left(\widehat{S_{ab}^\dagger X} \right)_t \beta_b^* \right. \\ &\quad \left. - \widehat{X}_t \lambda_a(t) \right\} dI_a(t) \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}(X) &= \sum_{a,b} \beta_a^*(t) \left[\sum_c S_{ac}^\dagger X S_{cb} - \delta_{ab} X \right] \beta_c(t) \\ &\quad - \frac{i}{\hbar} [X, H], \\ \lambda_a(t) &= \sum_b \left(\widehat{S_{ab}} \right)_t \beta_b(t) + \sum_b \left(\widehat{S_{ab}^\dagger} \right)_t \beta_b^* \end{aligned}$$

and

$$dI_a(t) = dY_a(t) - \lambda_a(t) dt.$$

The processes I_a are independent Wiener processes.

The corresponding stochastic master equation is then

$$d\varrho_t = \mathcal{D}\varrho_t dt + \sum_a \mathcal{H}_a[\varrho_t] dI_a(t),$$

with

$$\mathcal{D}\varrho = \sum_{a,b,c} S_{ab}\varrho S_{ac}^\dagger \beta_b \beta_c^* - \|\beta(t)\|^2 \varrho + \frac{i}{\hbar} [\varrho, H],$$

$$\mathcal{H}_a[\varrho] = S_{ab}\varrho \beta_b + \varrho S_{ab}^\dagger \beta_b^* - \lambda_a(t) \varrho.$$

We note that the mapping \mathcal{H}_a is nonlinear in ϱ since $\lambda_a(t) = \sum_b \text{tr} \left\{ \varrho_t \left(S_{ab} \beta_b(t) + S_{ab}^\dagger \beta_b^*(t) \right) \right\}$.

In the case where there is only one input (say the right side), we have $S = e^{i\theta(q)}$ and the stochastic master equation simplifies to

$$d\varrho_t = \left\{ e^{i\theta} \varrho_t e^{-i\theta} - \varrho_t + \frac{i}{\hbar} [\varrho_t, H] \right\} dt$$

$$+ \left\{ e^{i\theta} \beta(t) \varrho_t + \varrho_t e^{-i\theta} \beta(t)^* - \lambda(t) \varrho_t \right\} dI(t)$$

with $\lambda(t) = \text{tr} \left\{ \varrho_t \left(e^{i\theta} \beta(t) + e^{-i\theta} \beta^*(t) \right) \right\}$.

B. Photon Counting

Now set $\Lambda_{\text{in},a}(t) = \int_0^t b_a^*(s) b_a(s) ds$ be the input annihilation process, then we might aim to measure the fields

$$Y_a(t) = U(t)^\dagger [1 \otimes \Lambda_{\text{in},a}(t)] U(t)$$

which gives the output quadrature for $a = l, r$. The Y_a are (time-inhomogeneous) Poisson processes.

The filter equation for photon counting measurement is then, from equation (21) of [20],

$$d\widehat{X}_t = \widehat{\mathcal{L}X}_t dt$$

$$+ \sum_a \left\{ \frac{1}{\nu_a(t)} \sum_{b,c} (\widehat{S_{ab}^\dagger X S_{ac}})_t \beta_b^*(t) \beta_c(t) - \widehat{X}_t \right\} dJ_a(t)$$

where $\mathcal{L}(X)$ is as before, and

$$\nu_a(t) = \sum_{b,c} \left(\widehat{S_{ab}^\dagger S_{ac}} \right)_t \beta_b^*(t) \beta_c(t)$$

and

$$dJ_a(t) = dY_a(t) - \nu_a(t) dt.$$

From the unitarity of S , we have the identity

$$\sum_a \nu_a(t) = \|\beta(t)\|^2,$$

and this allows us to write

$$d\widehat{X}_t = \frac{1}{i\hbar} [\widehat{X}, H]_t dt$$

$$+ \sum_a \left\{ \frac{1}{\nu_a(t)} \sum_{b,c} (\widehat{S_{ab}^\dagger X S_{ac}})_t \beta_b^*(t) \beta_c(t) - \widehat{X}_t \right\} dY_a(t).$$

The corresponding stochastic master equation is now

$$d\varrho_t = \frac{1}{i\hbar} [H, \varrho_t] dt + \sum_a \mathcal{H}_a[\varrho_t] dY_a(t)$$

with

$$\mathcal{H}_a[\varrho] = \frac{1}{\nu_a(t)} \sum_{b,c} S_{ac} \varrho S_{ab}^\dagger \beta_b^*(t) \beta_c(t) - \varrho,$$

and $\nu_a(t) = \text{tr} \left\{ \varrho_t \sum_{b,c} S_{ab}^\dagger S_{ac} \right\} \beta_b^*(t) \beta_c(t)$.

This, of course, corresponds to a continuous Hamiltonian evolution under H , with jumps

$$\varrho \rightarrow \frac{1}{\nu_a(t)} \sum_{b,c} \beta_c(t) S_{ac} \varrho S_{ab}^\dagger \beta_b^*(t)$$

occurring at random times when we detect a photon at the right ($a = r$) or left ($a = l$) detector.

Again, this simplifies if we have only one input, and we find the stochastic master equation simplifies to

$$d\varrho_t = \left\{ e^{i\theta} \varrho_t e^{-i\theta} - \varrho_t + \frac{i}{\hbar} [\varrho_t, H] \right\} dt$$

$$+ \left\{ e^{i\theta} \varrho_t e^{-i\theta} - \varrho_t \right\} dJ(t)$$

$$= \frac{i}{\hbar} [\varrho_t, H] dt + \left\{ e^{i\theta} \varrho_t e^{-i\theta} - \varrho_t \right\} dY(t),$$

and $\lambda(t) = |\beta(t)|^2$.

ACKNOWLEDGMENTS

The author wishes to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Quantum Control Engineering* where work on this paper was completed. He acknowledges several fruitful discussions with Howard Wiseman, Andrew Doherty, Ramon van Handel, Luc Bouten, Hendra Nurdin, Jake Taylor and Matthew James.

Appendix A: The Holevo Time-Ordering and

We now justify the limit of $\tilde{U}(t)$ as a Holevo time-ordered exponential

$$\tilde{U}^{(n)}(t + \tau, t) = \tilde{U}^{(n)}(t + \tau) \tilde{U}^{(n)}(t)^\dagger = \sum_{r=0}^n (-i)^r \int_{t+\tau \geq t_r \geq \dots \geq t_1 \geq 0} \tilde{H}^{(n)}(t_r) \dots \tilde{H}^{(n)}(t_1) dt_r \dots dt_1. \quad (\text{A1})$$

For a fixed n , we partition the interval $[t, t + \tau]$ with grid points $\sigma_j = t + \frac{j\tau}{N}$ for $j = 1, \dots, N$ where $N = \frac{n\tau}{2c}$. Each of the first N terms in the expansion of (A1) may be approximated as the discrete sums

$$(-i)^r \sum_{N \geq j_r \geq \dots \geq j_1 \geq 1} \tilde{H}^{(n)}(\sigma_{j_r}) \dots \tilde{H}^{(n)}(\sigma_{j_1}). \quad (\text{A2})$$

Now we compare this to the exponential $\exp\{-i \sum_{j=1}^N \tilde{H}^{(n)}(\sigma_j)\}$ which we may likewise expand leading to the r th term

$$\frac{(-i)^r}{r!} \sum_{j_r, \dots, j_1=1}^N \tilde{H}^{(n)}(\sigma_{j_r}) \dots \tilde{H}^{(n)}(\sigma_{j_1}) = \frac{(-i)^r}{r!} \sum_{j_r, \dots, j_1=1}^N E_{\alpha_{j_r} \beta_{j_r}} \dots E_{\alpha_{j_1} \beta_{j_1}} \otimes \tilde{\lambda}_{\alpha_{j_r} \beta_{j_r}}^{(n)}(\sigma_{j_r}) \dots \tilde{\lambda}_{\alpha_{j_1} \beta_{j_1}}^{(n)}(\sigma_{j_1}). \quad (\text{A3})$$

The crucial observation is that $\tilde{\lambda}_{\alpha\beta}^{(n)}(t)$ and $\tilde{\lambda}_{\mu\nu}^{(n)}(s)$ will commute whenever $|t - s| \geq \frac{c}{n}$, and so for $r \leq N$ the various $\tilde{\lambda}^{(n)}$ terms commute in the expression above. Therefore for a fixed set of indices $\alpha_{j_r}, \beta_{j_r}, \dots, \alpha_{j_1}, \beta_{j_1}$

we may reorder the $\tilde{\lambda}^{(n)}$'s in (A3) in $r!$ equivalent ways, thereby recovering (A2). Therefore the first N terms of series expansion (A1) agree with the first N terms of $\exp\{-i \sum_{j=1}^N \tilde{H}^{(n)}(\sigma_j)\}$.

We then take the limit $n \rightarrow \infty$ and $\tau \rightarrow 0$ to obtain in principle the Holevo time-ordered exponential (9).

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